Journal of Pedagogical Research
Onlinefirst Article, 2024
https://doi.org/10.33902/JPR.202427803

*Research Article*

# The process of pre-service mathematics teachers reaching spatial visualisation generalisations

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The study aims to define the processes of pre-service mathematics teachers in reaching spatial visualisation generalisations within the context of drawing surface nets of solids. Two theories, Polya's problem-solving steps and novice-to-expert problem-solving schemas, were used as reference frameworks to describe the participants' spatial visualisation generalisation processes. The research methodology employed in this study was a qualitative theory-testing case study design, in which hypotheses based on those two theories were generated and tested. The sample consisted of 44 participants who completed low- and high-complexity spatial visualisation drawing tasks and attended task-based interviews. The findings obtained by qualitative data and verified with quantitative data revealed participants' problem-solving processes involved a series of steps, including the creation of a mental representation of the problem situation (comprehending the configuration and the requirements of the task), devising an appropriate strategy to unfold the surface of the solid, implementing the strategy enabling to draw one of the nets of the solid, and finally evaluating the accuracy of the net drawing representing the opened surface of the solid. The participants' three developmental spatial visualisation stages (novice, competent, and expert) were identified based on their spatial visualisation problem-solving performances in low- and high-complexity tasks.

Keywords: Generalisation of spatial visualisation; Pre-service mathematics teachers; Problem-solving processes and schemas of novice-to-expert; Theory-testing using case studies

Article History: Submitted 17 April 2024; Revised 7 October 2024; Published online 30 October 2024

## 1. Introduction

If an organism is rewarded for a precise response to a specific stimulus, it will probably generate a comparable reaction when confronted with stimuli that bear similar characteristics (Cheng & Spetch, 2002). This phenomenon is known as stimulus generalisation (Cheng & Spetch, 2002). In computer games, players controlling the character's movement within a virtual space can advance to the next level by making generalisations of a sequence of stimuli encountered throughout a level. The literature on spatial skills highlights an intriguing finding: novice learners often adopt trial-and-error methods to solve spatial tasks, while those with expertise rely on spatial generalisations. Despite the extensive research on spatial skills, spatial generalisations have received relatively scant attention in the literature. That underscores the need to comprehensively explore spatial generalisations' role in developing spatial skills. In their recent study, Manivannan

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**How to cite:** Kurban, F. & Yanık, H. B. (2024). The process of pre-service mathematics teachers reaching spatial visualisation generalisations. *Journal of Pedagogical Research*. Advance online publication. https://doi.org/10.33902/JPR.202427803

et al. (2022) proposed a generalisation process regarding spatial orientation based on data about mapping spatial relations in an area by human subjects' distorted, incomplete, complete, or accurate drawings with or without metric features. According to Barbosa and Vale's research (2015), the process of generalising shape patterns involves several steps, such as identifying individual sets of elements that constitute the initial figure, determining intersecting sub-sets of these element sets, and recognising the involvement of certain elements more than once and then removing them. The current study describes the generalisation of pre-service teachers' spatial visualisation process in the context of drawing the solids' nets, which has not been previously explored in the realm of spatial generalisation research.

Spatial visualisation is defined as the cognitive ability of the human brain to perceive, generate, and manipulate two- or three-dimensional (2D or 3D) models, virtual images, or objects and store them in long-term memory. This ability is fundamental to many fields, including engineering, architecture, and mathematics. To evaluate learners' spatial visualisation skills, professionals use tasks that require visualising the images obtained after rotating or cutting shapes or solids or unfolding the surfaces of solids (Markopoulos et al., 2015). Tasks that involve obtaining surface development of solids and vice versa are commonly known as spatial visualisation tasks. However, dealing with such tasks requires the activation and utilisation of various cognitive skills, including spatial visualisation, spatial reasoning, and geometric thinking. Spatial visualisation and reasoning enable us to mentally represent the spatial relationships between solids and their components and the changes that occur when certain sequential transformations are applied to them. These processes of representation and transformation are intricately linked to mathematics, as highlighted in the research conducted by Lowrie et al. (2020).

Those who perform well on spatial tasks also tend to perform well on mathematics ability tests (Lowrie et al., 2017). These findings, along with evidence, suggest the same neural pathways are activated during spatial and numerical tasks, which have resulted in an increased interest in developing spatial skills to improve math performance and enhance the learning experience for individuals at all levels (Lowrie et al., 2017; Otumfuor & Carr, 2017). Research suggests that educators with advanced spatial skills are better equipped to communicate mathematical concepts to their students by utilising visual and spatial representations in their instructional methods (Otumfuor & Carr, 2017). Students' endeavours to enhance their spatial and mathematical thinking abilities may inadvertently be impeded by mathematics teachers who lack an understanding of the intercourse between spatial and mathematical teaching approaches and have not developed adequate proficiency in spatial tasks (Otumfuor & Carr, 2017). Acquiring insightful information on the levels of expertise and spatial generalisation processes of pre-service teachers is highly valuable for mathematics teacher training programs. This data can be utilised to establish a deeper connection between mathematics and spatial content and to integrate theories on spatial skills in the pedagogy of mathematics instruction. By adopting spatial-pedagogical approaches, it is possible to train expert teachers capable of effectively teaching mathematics to their students.

The present study aims to define the processes of pre-service mathematics teachers in reaching spatial generalisations within the context of drawing surface nets of solids. The theoretical frameworks concerning the construction of expert problem-solving schemas serve as a guide to examine this process. In alignment with the research's overarching aim, the two questions were sought to answer:

RQ 1) What spatial visualisation problem-solving steps do participants experience throughout their problem-solving processes?

RQ 2) What developmental stages do pre-service mathematics teachers' spatial visualisation processes follow?

### 1.1. Theoretical Considerations

A problem-solving schema serves as a well-organized framework or cognitive model that develops and adapts through an individual's experiences with mental, cognitive, spatial, and mathematical processes (Marshall, 1995). The schema allows to organize one's experiences in such a way that a new similar experience can easily be recognized and dealt with successfully (Cadez & Kolar, 2015). A new experience is assimilated into an existing schema, or an existing schema is modified to accommodate the new experience (Cadez & Kolar, 2015). Generalisation, allowing the individual to abstract solutions from specific cases and apply them to similar situations, contributes to the development of problem-solving schemas by enabling the individual to tackle new problems by applying previously learned strategies (Kalyuga & Hanham, 2011). During the generalisation of the problem-solving process, learners encounter several challenges, caused from inadequate prior knowledge, relying on specific examples or memorized procedures, the complexity of new problems, misunderstandings and misconceptions and make several generalisation mistakes, compassing overgeneralisation, inadequate generalisation, focus on surface features, misinterpretation of patterns and failure to reflect on errors. The implementation of schema-based education, which considers the challenges encountered by students and encompasses effective, level-appropriate instructions, can be realized by understanding the nature of their learning and problem-solving processes (Jung et al., 2022).

Problem-solving, a structured process, is not just about finding the correct answer but also about the thinking process that fosters a profound comprehension of mathematical concepts through a systematic approach to learning. Polya (2004) framed learners' problem-solving processes in a four-step systematic and organized model, including (a) understanding the problem, (b) developing strategies, (c) applying the chosen strategy and (d) revising or evaluating the solution. Polya (2004) eloquently articulated the dynamic nature of problem-solving, emphasizing the evolution of learners' perception of a problem as they progress toward finding a solution. He emphasized that learners' initial understanding of the problem often contains gaps, yet becomes enriched and transformed as they navigate through the various stages of problem-solving. Polya underscored the pivotal role of each stage in the successful resolution of a problem, shedding light on the intricate and evolving process of problem-solving (Turkoglu & Yalcınalp, 2024).

The need to teach learners how to develop expertise in problem-solving is guided to investigate the behavioral patterns of novice and expert problem-solvers (Ericsson, 2006; Ngu & Phan, 2022). The development of problem-solving schemas is a key factor in the transition from novice to expert problem-solving abilities. Experts with a superior ability to process spatial information utilise their vast knowledge resources and activate their problem-solving schemas, classified in terms of their solutions' deep and characteristic features, to address complex problems (Elio & Scharf, 1990; Persky & Robinson, 2017). Experts' problem-solving process involves a set of distinct steps, including the construction of a mental representation of the problem, exploration for suitable problem-solving strategies and procedures in long-term memory, application of the relevant strategy and procedures after recalling the necessary information into working memory, evaluation of the problem-solving process and solution, and categorisation and storage the problem situation and its solution in long-term memory (Newell & Simon, 1972; Nokes et al., 2010). Repeating these stages, although not sequentially, ensures the building of robust and categorical problem-solving schemas (Nokes et al., 2010). In contrast, novices with limited experience tend to rely on prototypes that categorise problems based on their superficial characteristics (Elio & Scharf, 1990). Individuals can acquire mastery in categorical tasks by enhancing their conceptual and procedural knowledge, acquiring strategies that enable them to deal with these tasks effectively and applying these strategies. The progression from a novice stage to an expert stage is not a direct process; learners must undergo various stages (Dreyfus & Dreyfus, 2005). The situation is similar for learners who reach spatial generalisations.

Experts' approaches differ significantly from novices' in various problem-solving steps. Novice problem-solvers rely solely on the information and relationships explicitly mentioned in the task to form mental representations of the problem situation. Most of their problem-solving errors result from incorrect mental representations of the problem (Cifarelli, 1998; Lewis, 1989; Sutherland, 2002). Mental representations are temporary cognitive structures that model the problem situation automatically constructed by the problem-solver (Björklund, 2013). These include the conceptual structures implicit in the problem situation and the strategies and procedures mandated for the solution (Nokes et al., 2010; Sutherland, 2002). While experts quickly connect with these structures when solving problems, novices cannot easily link with other previously learned knowledge when solving a problem, and the strategies they use when solving the problem are intuitive (Nokes et al., 2010; Persky & Robinson, 2017). In problem-solving processes, novices may act based on their preconceptions and biases, leading to inappropriate decisions and a higher likelihood of errors and negligence (Persky & Robinson, 2017). On the other hand, experts employ appropriate domain-specific strategies, reflect on their procedures while implementing solutions, and can identify and correct their mistakes, including those that stem from misrepresentations of the problem situation in their minds (Nokes et al., 2010). The study of novice-expert problem-solving processes and schemas has long been a topic of interest among researchers. However, there has been a lack of discussion on learners' processes of reaching spatial generalisations from this perspective. This is a considerable research area that deserves further exploration and investigation.

## 2. Method

The present study employed a qualitative theory-testing case study design based on a deductive approach and pattern-matching analysis. Qualitative deduction enables the researcher to scrutinise existing theories, gather data to support them, and affirm or disprove them based on the obtained data (Creswell, 2014). Additionally, pattern matching enables the comparison of the proposition patterns derived from the theories to be tested with the patterns of observed cases, thereby facilitating the identification of connections between them (Bitektine, 2008; Sinkovics, 2018; Vargas-Bianchi, 2020). The two theories were used as reference frames to describe the participants' processes of reaching spatial generalisation in the context of drawing the surface nets of solids. These theories were Polya's problem-solving steps and novice-expert problem-solving schemas, explicitly addressing the novice, competent and expert problem-solving behaviours. The theories made it possible to predict what will happen in the phenomenon under study (Lokke & Sorensen, 2014). The tasks of drawing the surface nets of solids were treated as problem situations in the study, and the behavioural patterns of the participants during the problem-solving process were investigated within the context of these two theoretical frameworks; thus, it was tried to define the spatial generalisation processes of learners. The theories were extended to participants' behavioural patterns in problem-solving steps and their performances on spatial visualisation tasks to describe the processes of reaching spatial generalisations. The purpose of theory testing is not only to test the compatibility of collected data with theories but also to refine, develop, and expand these theories' boundaries (Bhattacherjee, 2012). Hence, how participants reached spatial generalisations was defined according to the behavioural patterns exhibited in the spatial problem-solving steps and the characteristics of the participants' performances in spatial tasks, which point out the development stages of their problem-solving schemas.

### 2.1. Participants

The study participants were determined using purposeful and identical sampling techniques (Cresswell, 2013; Johnson & Christensen, 2020). The purposeful sampling allows for forming a sample that can provide detailed and in-depth information about the studied domain (Mertens, 2020; Sargeant, 2012). Considering that those with low spatial visualisation skills cannot reach spatial generalisations, while those with high skills are most likely to do so, this study aimed to create a sample with spatial visualisation performance diversity using a purposeful sampling technique. The sample was chosen from second-grade students who continue a mathematics teacher training program, and all took part in the Purdue Spatial Visualization Test [PSVT]. The test includes tasks that require individuals to match surface nets, or 2D shape patterns, with appropriate solids. The data gathered from pre-service mathematics teachers who showed low (15), medium (18), and high performance (11) in this test were likely to describe a better understanding of the process of reaching spatial visualisation generalisations.

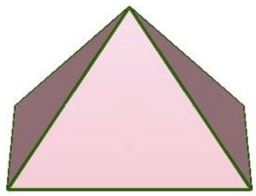
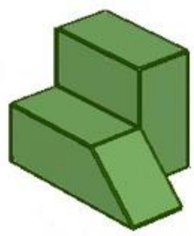
The study sample consisted of pre-service teachers who fulfilled low- and high-complexity drawing tasks and participated in task-based interviews provided to preserve the diversity of spatial visualisation performance. That refers to the identical sampling technique in which qualitative and quantitative data are gathered from the same participants (Johnson & Christensen, 2020; Onwuegbuzie & Collins, 2007). It was anticipated that conducting interviews with the entire group, 44 participants (42 female and two male), would facilitate the elucidation of their process for arriving at spatial generalisations. A systematic review analysis in qualitative research suggests that the sample size required to be interviewed typically range from 9 to 17 to achieve data saturation (Hennink & Kaiser, 2022). However, for more complex studies addressing broader themes, a larger sample of 20 to 40 interviews may be necessary (Hagaman & Wutich, 2017). Data saturation is reached when no additional issues or insights are identified, indicating that the collected qualitative data adequately represents the full breadth of people's experiences (Hennink & Kaiser, 2022). Because of the recommendations for the sample size in the qualitative analysis literature range from 5 to 60 interviews, the number of participants interviewed in this study was considered sufficient (Constantinou et al., 2017; Guest et al., 2006).

In addition to the above features, the participants received advanced mathematics, geometry, and teacher training courses at the time of the study. Furthermore, they had preliminary knowledge and experience with the nets of basic solids due to compulsory education exposure in the same country spanning elementary, middle, and high school years.

### 2.2. Data Collection Tools and Procedures

This study relied on drawing tasks, task-based interviews, and field notes as sources for data collection. Participants completed five low-complexity tasks [LCT] and five high-complexity tasks [HCT], requiring them to draw surface nets of solids on checkered paper by considering their geometric properties (see Table 1). While LCTs include prisms and pyramids (basic solids), whose faces are convex polygons, HCTs contain composite solids, formed from two different face types: (a) convex faces, inherited from their constituent-basic solids and (b) concave faces, formed from the unification of the two faces, each belonging to different basic solids or the subtraction of one face from the other one.

Table 1
*LCT and HCT examples and problem situation*

| **Rectangular Pyramid** | **The Composite Solid** |
|---|---|
| (a)  | (b)  |

| **Problem Situation for the Tasks** |
|---|
| First, imagine that you unfold the given solid along the fold lines to its sub-base such that its inner surface is visible. Contruct the net, that you imagined, on the checkered paper by taking account the geometric properties of the solid and its faces, and utilizing the suitable geometric tools (e.g., ruler, protractor). |

Based on the participants' drawings, the researcher generated hypotheses regarding participants' developmental stages of problem-solving schemes and their challenges in specific problem-solving steps. Theoretical considerations regarding Polya's problem-solving algorithms and novice-expert problem-solving schemes guided the generation of these hypotheses. For instance, a drawing with considerable errors was matched with a novice stage, a drawing without errors was associated with an expert stage, and a drawing with only a few errors was mapped with a competency stage.

In order to verify or refute the hypothesis under consideration, a series of interview questions (e.g., Can you describe the solid? What geometric shapes are the solid's surface formed? Which one was the reference face or base of your net drawing? How and in what order did you unfold the solid's faces? How did you reflect the geometric features of the faces in your net drawing? How did you close your net drawing? Do you recognise the solid?) were formulated in a way compatible with Polya's theoretical framework to reveal the behavioural patterns of participants in different problem-solving steps. Upon completion of the drawings, task-based interviews were conducted with each participant to test the produced hypotheses and to gather data regarding the participants' thought processes while they were solving spatial visualisation tasks. One of the two purposes of the interviews was to identify the relationship between the challenges encountered by the participants in their problem-solving steps and their erroneous thinking ways. The other purpose was to gather data to characterise the behavioural patterns of the participants at distinct stages of proficiency, namely, novice, competent, or expert, for each of the tasks in both LCT and HCT. Task-based interviews are a commonly used tool in mathematics education to gather information regarding the current and evolving mathematical knowledge and problem-solving behaviours of an individual or a group of learners (Maher & Sigley, 2020).

Participants' verbal and visual expressions (e.g. hand gestures) were meticulously recorded during the interviews using audio and video recorders, while detailed field notes were taken for further analysis. Before the interviews, participants were asked to explain their thoughts and reasoning processes by connecting the given solids and their drawings while performing the tasks. Although the interview questions directed to the participants were prepared based on problem-solving and novice-expert literature, their unexpected explanations during the interviews caused to be made specific changes to some of those questions (Maher & Sigley, 2014; Mejía-Ramos & Weber, 2020).

### 2.3. Data Analysis

Data obtained from participants' drawings, interviews, and field notes were analysed using the qualitative theory testing method, which includes deductive and pattern-matching approaches. The data were analysed in light of two theoretical frameworks: Polya's (1957) problem-solving steps and the characteristics of problem-solving schemas used by learners at different stages, including novice, competent, and expert. The deductive approach uses a structure or predetermined framework to analyse data (Burnard et al., 2008). Data collection and analysis were ongoing processes carried out simultaneously. In order to analyse the participants' process of reaching spatial generalisations, a data analysis diagram was created that was compatible with these two theories and shaped by hypotheses that were tested and verified as data were collected.

The hypotheses were produced by the researcher who connected those theories with his/her own experiences. The validity of those produced hypotheses (predicted theoretical patterns) was tested according to whether the cases predicted or foresaw to occur match with the observed behaviour pattern. In pattern matching, the researcher compares these hypotheses with the observed patterns to decide their validity and attempts to show that the theory overlaps with the studied context (Bitektine, 2008; Sinkovics, 2018; Vargas-Bianchi, 2020). The researcher rejected the hypotheses when they did not match the observed patterns. The iterative process of formulating hypotheses, gathering empirical data, and testing the plausibility of these hypotheses continued until the two theoretical frameworks explained all the observed problem-solving behavioural patterns of participants.

The data analysis table includes the validated hypotheses regarding knowledge and skills that must be possessed to successfully perform any spatial visualisation tasks without making any errors in each problem-solving step: understanding the problem, developing a strategy, selecting and applying the appropriate strategy and evaluating the solution. The table also includes the validated hypotheses concerning the participants' behavioural patterns who were in novice, competent and expert stages, defined about the particular cases: whether participants experienced challenges in any of the problem-solving steps, at what problem-solving step participants experienced difficulty, whether they made any errors in their drawings and whether their drawing errors were minor or major.

The procedures carried out to ensure the validity and reliability of the study were: (a) the data analysis, the production of the research hypotheses, and the preparation of the interview questions were grounded on the two theories on which many studies have been conducted, (b) a data analysis scheme was created according to the derived hypotheses and given its final form according to the validated hypotheses, (c) the collected data were analyzed simultaneously and separately by the researcher and another experienced mathematics educator, and the inconsistencies between them were eliminated through discussions, and (d) the study sample included as many participants as possible.

The data analysis diagram (see Table 2) was organized according to the tested and verified hypotheses used to analyze the participants' process of reaching spatial generalisations in the context of the tasks of drawing the solids' surface nets. The codes in the diagram address the behavioural patterns exhibited in the problem-solving steps and characteristics of problem solvers at different developmental stages.

## 3. Results

Remarkable findings were achieved upon analyzing the gathered data concerning participants' processes of reaching spatial generalisations in the context of drawing the surface nets of solids. The tested and verified hypotheses showed that the participants' processes of reaching spatial generalisations regarding the solution of spatial visualization tasks were compatible with the two theoretical frameworks: Polya's problem-solving algorithm and novice-expert problem-solving schemas. Findings indicate that participants' ability to reach generalisations in solving spatial visualization tasks is linked to their problem-solving processes and schemas.

### 3.1. Participants' Problem-solving Processes

From the verified hypotheses, it was found that participants' behavioural patterns while solving spatial visualisation tasks were consistent with the problem-solving steps defined by Polya. These are understanding the configuration of the solid and the task requirements (understanding the problem), developing a strategy to open the solid (strategy development), implementing the chosen strategy that gives the solid's net drawing (implementing the chosen strategy), and evaluating the resulting net drawing (evaluating the solution). The findings reached through tested and verified hypotheses compatible with the theoretical framework regarding the solution process of spatial visualisation tasks consisted of the behavioural patterns of the participants describing each problem-solving step, numerical data related to these behavioural patterns, sample participant drawings and the parts taken from the participant interviews.

Table 2

*Data analysis diagram formed from produced hypotheses and shaped according to verified hypotheses including learners' problem-solving steps and the developmental stages of problem-solving schemas*

| *Problem-solving Steps for Spatial Visualisation Tasks* | |
|---|---|
| Understanding the task | Understanding the configuration of the solid:<br>Imagining the solid accurately from its 2D representation<br>Deducing the geometric properties of the solid's faces accurately from its 2D representation<br>Owning geometric concept knowledge regarding the properties of solids and their components |
| | Understanding what is asked in the task:<br>Having broad conceptual knowledge about the surface and nets of a polyhedron<br>Using existing knowledge about how to open the given solid to its base |
| | Using existing knowledge of how familiar solids unfold to determine how to unfold the surfaces of complex solids<br>Taking in the account the information presented in the task regarding how the solids should be opened |
| Developing a strategy | Inability to develop any strategy or develop a wrong strategy to open the surface net of the given solid<br>Leveraging a known prototypical strategy to open the surface of the given solid<br>Developing a new strategy to open the surface of a given composite solid by combining and coordinating the prototype strategies of the basic solids that form it.<br>Developing a relatively more advanced strategy for opening the surface of the given solid |
| Implementing the strategy | Using definitions and properties of the shapes on the solid's faces, while constructing the solid's net on the paper (Conceptual Knowledge)<br>Reflecting the face shapes of the solid onto paper by taking advantage of squared paper's properties, using geometric tools and following geometric construction rules and procedures (Procedural Knowledge)<br>Reflecting particular spatial and geometric relations between edges and angles of adjacent faces of the solid into that solid's net drawing (Spatial Skills) |
| Evaluating the solution | Controlling whether any errors were made in any of the problem-solving steps while drawing the solid's net<br>Making drawing by considering the folding or closing process of the solid's net<br>Making drawing by considering that the pairs of edges that will join when the solid's net is closed must be of equal length (Geometrical and Spatial Reasoning) |
| *Spatial visualization stages* | |
| Novice | Having challenges in understanding the spatial task, which cause the solution to be radically flawed |
| Competent | Being able to understand the task to a great extent and develop a correct strategy to open the surface of the solid, but having some challenges in applying the strategy, making minor mistakes in the solid's net drawing, and not consciously and carefully controlling whether the accurate net drawing of the solid was made or not. |
| Expert | Having advanced problem-solving schemas that ensured to make the accurate net drawing of the solid including to draw the edge pairs in equal lengths, which will join when the solid's net is closed |

*3.1.1. Theoretical structure regarding the step "understanding the task"*

A precise mental representation of a given solid needs both spatial (e.g., comprehending a 3D configuration from its 2D representation and the ability to manipulate and rotate the solid holistically in mind), and geometric understanding (e.g., containing knowledge of various concepts such as the solid and its surface, nets, components and properties (such as faces, edges, and corners)). Understanding the configuration of a solid demands the concurrent usage of a range of spatial and conceptual processes, in addition to the coordination among them. From the study data detected processes were distinguishing a basic or an unfamiliar solid's components and knowing their geometric properties (for example, the side faces of a triangular prism are rectangular), comparing the edge lengths of the solid mentally and denoting these lengths in the same unit, comprehending a composite solid's geometric properties, which is formed after combining two or three basic solids and inferring and visualising that composite solid's and its faces' geometric properties and its invisible faces. Most competent participants did not represent some faces of the solids in their drawings, and they inaccurately reflected at least one of the concave faces of the composite solids. The study's findings indicated that comprehension of instructions necessitates possessing conceptual knowledge about solids' nets, as well as considering the information presented in the task regarding unfolding the given solid and visualising the unfolding process. Unfolding the solid's surface permits its entire interior or exterior surface to be visible, utilising either the bottom base or one of the side faces as a reference point.

*3.1.2. Participants' behavioural patterns compatible with the theoretical structure regarding the step "understanding the task"*

More than half of the participants (28) could accurately create a mental representation of the pentagonal prism from its 2D representation. However, a significant number of participants (16) identified it as a right trapezoid because they had difficulty perceiving its 3D configuration. Similar behavioural patterns were observed in tasks 6, 7, and 10, being of compound solids. Out of all participants, only 5 made an error in identifying the position of the cube's bottom base in Table 3, stating that it is precisely in the middle of the upper surface of the square prism, while the remaining recognized its placement correctly. The 34 participants knew that the side faces of the triangular prism represented on the 2D plane were rectangular, unlike a significant number of participants (8) who perceived the rectangular faces of the prism to be parallelograms. Likewise, some participants perceived the solids' oblique faces in the 5th, 7th and 8th tasks as parallelograms, though they are rectangular (see Table 3).

Of the participants who had misconceptions about what the nets of the solids, overlapped the faces while opening the solid (e.g., 5 participants in the 6th task and 4 in the 10th task), and/or did not get the specific faces of the solid to touch the base plane completely— left those faces inclined (e.g., 4 participants in the 4th task and 3 in the 10th task). Participants who did not understand what was asked in the task either took another face of the solid as a reference instead of its base (e.g., 4 participants in the 2nd task and 2 participants in the 5th task) or opened the surface of the solid such a way that its entire outer surface or some parts of its outer surface were visible instead of its inner surface (e.g., 5 participants in the 7th task, 3 in the 8th task and 3 in the 10th task). Considering all HCTs, in the 8th task, ten participants could accurately represent the faces created concave faces that arose from combining basic solids in their net drawings. In other tasks, this number was significantly lower; that is, most participants represented those faces incorrectly or did not reflect these faces in their net drawings. A particular number of the competent participants drew a whole square or rectangle for one of the concave faces in the HCT tasks (e.g., 7 participants in the 6th, 4 in the 9th task and 4 in the 10th task). Apart from these, 18 participants who had misconceptions regarding the properties of composite solids or their nets believed that to obtain the net of a composite solid, they needed to separately open the basic solids that form the composite solid in at least one of their drawings (see Table 4).

Table 3
*Example behavioural patterns exhibited by the participants in step of understanding the given solid's configuration*

**Understanding the solid's configuration:**

In the 6th task, while Ebrar incorrectly represented the compound solid in his mind and could not draw its net, Hafize made the correct mental representation of the same solid. The part taken from the interviews and two images depicting their thoughts reflect the mental representations of the participants.

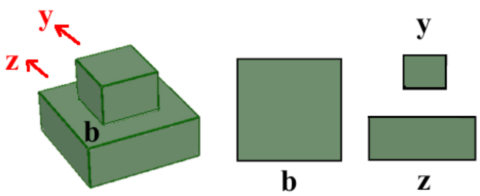
**Interview parts:**

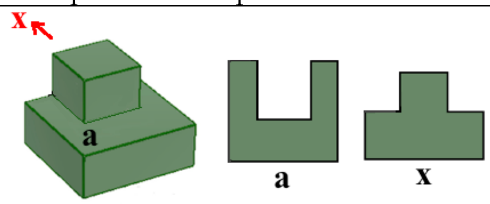
Ebrar: I could not draw this because I did not understand...I were not able to decide what was the form of this solid, so I could not draw...
Researcher: How is the form of this solid according to you?
Ebrar: There is a square here, full a square (10). I thought that they put a cube in the middle of it... I did not understand what kind of solid it was, so I did not make its net drawing.

Hafize: ...the back faces of this cube and this prism are joined. (There is) a thin rectangle at the back and a square just above it...it looks like the letter T, but it is an upside-down T.

**Mental Representations:**

| Erroneous mental representation of Ebrar<br>The composite solid represented in the 6th task | Accurate mental representation of Hafize<br>The composite solid represented in the 6 th task |
|---|---|




**Understanding the solid's configuration**

From its 2D representation, Banu thought that two of the rectangular faces of the triangular prism were parallelogram-shaped, while Pelin correctly represented those faces in her mind. Parts of the interviews with Banu and Pelin and participant drawings confirm this finding.

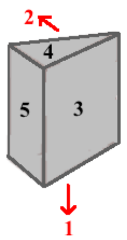
**Interview parts:**

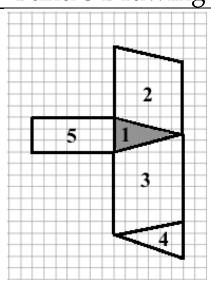
Researcher: Do you recognize the object below?
Banu: Yes, I know it, a triangular prism.
Researcher: What shapes are there on the surface of this object?
Banu: There are parallelograms (2 and 3), rectangles (5), triangles, and isosceles triangles (1 and 4), so that's it.

Pelin: Triangular prism. Its base (1) and top one (4) are triangles; this (3), that one (5) and the one at the back (2) are each a rectangle (showing the faces by touching them with a pencil to the solid's 2D representation).

**Drawings:**

| The solid in the 2nd task | Banu's Drawing | Pelin's Drawing |
|---|---|---|



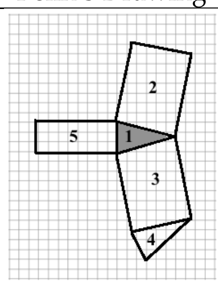



Table 4
*Example behavioural patterns exhibited by the participants in step of understanding what was asked in the task*

**Understanding what was asked in the task:**

While Sema took the bottom face of the solid as the base, Nazlı took the solid's face labelled with two (2) as the base. Nazlı and Sevda's drawings and interview parts confirm this finding.

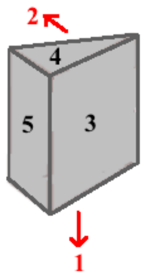
**Interview parts:**

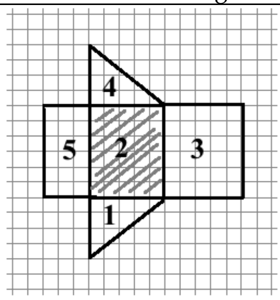
Researcher: Which face did you take as the base?
Nazli: The unseen face in the background.

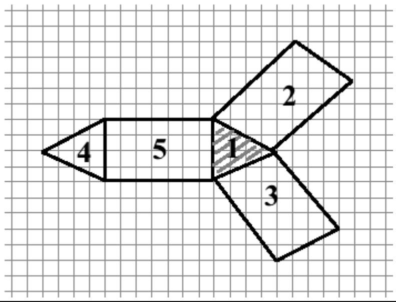
Sema:... (shading) This is my bottom base (1)...These are the side faces (2, 3, and 5). This is my top face (4); I think it is the same as my bottom base.

**Drawings:**

| The solid in the 2nd task | Nazlı's drawing | Sema's drawing |
|---|---|---|




**Understanding what was asked in the task:**

Since Betul had misconceptions about the composite solid's surface and nets, she left one of the faces without lying it entirely on the ground plane. Sara, with the necessary knowledge and skills, made the correct net drawing of the same solid.

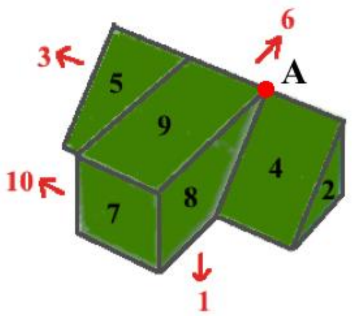
**Interview parts:**

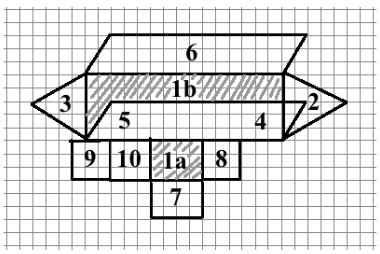
Researcher: I understand that these squares (1a, 7, 8, 9 and 10) form the cube. I don't understand how that face (face labeled 5 and 4) stands? Is it on the ground plane?
Betul: It stands like this (s/he stands her/his hand perpendicular to the table plane)
Researcher: Does it stand in the air?
Betul: Yes, it actually stands perpendicular to the ground plane.

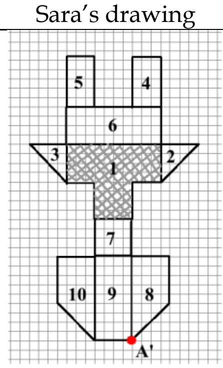
Sara: I thought that the bottom base (1) was whole.
Researcher: What do you mean by using the word, whole?
Sara: It is like these two solids (trapezoidal prism and triangular prism) are glued together, but they have a common base (1). I thought that the bottom bases (the base of a trapezoidal prism is square, and the base of a triangular prism is rectangular) were one piece, and I drew that way. I fixed the bottom base. Then, I laid its side faces down (showing the faces 2 and 3). Then, I connected these two faces (4 and 5), rectangular in shape, to the behind face (6), which face we can not see. For example, if we take this as the front face (7), this will be the back face (6). I unfolded the back face (6) towards the back and connected these two (4 and 5) there...I saw a square face (7) over there. I laid it forward. Then, there are three faces (8, 9 and 10), connected to that square face. After I laid the square one (7) down, I thought that this rectangular face (9) came forward with the square face. Then, there is a trapezoid (8) here. I thought that if we call this corner of the trapezoid as corner A, this corner A also lies with them. Corner A will also come forward. For example, let's say base A will be here. I drew the trapezoids (8 and 10) accordingly.

**Drawings:**

| The composite solid in the 10th task | Betul's drawing | Sara's drawing |
|---|---|---|



### *3.1.3. Theoretical structure regarding the step "strategy development"*

One technique to open a solid, making its entire inner surface visible, is to rotate its faces along the fold lines at certain angles in a specific order and lay down all the faces of that solid to the base plane. The net is formed if the faces of the solid are brought to the same base plane by applying rotational movements. Learners can develop many simple or complex unfolding strategies to open a solid by using this technique that necessitates visualizing the movement of the faces. The findings demonstrated that participants tend to use familiar prototypical strategies (such as opening the solid's side faces around its base) that they learned during their primary, middle, and high school years to open the surfaces of low-complexity solids. Likewise, most participants who successfully produced net drawings for high-complexity solids developed strategies that facilitated opening the surface of the composite solids by coordinating the prototype strategies of basic solids. In both LCT and HCT, some participants developed distinct or more sophisticated strategies to open the solids. Apart from these, developing a wrong strategy to open the solid's surface or failure to develop any strategy was less common in LCT but more common in HCT.

### *3.1.4. Participants' behavioural patterns compatible with the theoretical structure regarding the step "strategy development"*

In the LCT, participants generally adopted a prototype unfolding strategy through which the solid's side faces surrounded its base (e.g., 34 participants in the 1st task, 14 in the 2nd and 36 in the 4th task). Slightly more than half of the participants opened the triangular prism so that its rectangular faces were arranged side by side. These two are the most preferred unfolding strategies in mathematics teaching for opening the basic solids' surfaces or making those solids' net drawings in elementary, middle and high school years. Moreover, 5 participants in the 1st, 4 in the 4th and 3 in the 5th task used a sophisticated strategy to unfold the solid's surface. However, 4 participants in the 2nd task, 3 in 3th task, 4 in the 4th task, and 16 in the 5th task developed an incorrect strategy or could not create any strategy to open the solid's surface (see Table 5).

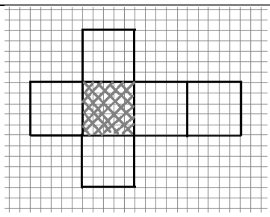
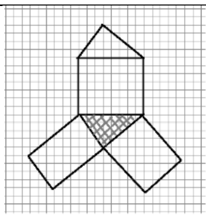
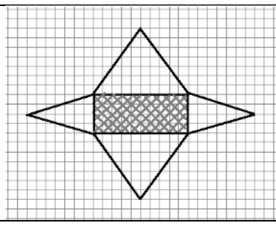
In the HCT, a significant number of participants (e.g., 13 participants in the 7th task, 22 in the 8th task) developed a strategy to open the composite solid's surface by coordinating the basic solids' prototypic unfolding strategies (constituents of the composite solid). Unlike that, 27 participants in the 6th task, 30 in the 7th task, 20 in the 8th task, 17 in the 9th task and 22 in the 10th task were unable to open the composite solid's surface due to either developing an incorrect strategy or failing to develop a strategy. In addition, only one participant developed advanced strategies for the 5th and 7th tasks, while the number of participants for the 8th task was 2.

Table 5
*Example behavioural patterns exhibited by the participants in step of developing a strategy*

| **Developing strategy:** |
|---|
| Tuna used the unfolding strategy, enabling the placement of the side faces around the bottom base to obtain all first-category solids' nets. Tuna's drawings for the net of the cube, triangular prism, and rectangular pyramid and the interview confirm this finding. |
| **Interview parts:** |
| Tuna: I always used the same method. At first, I couldn't understand the location of the top and bottom base shapes, but then I fixed the bottom base and opened the adjacent faces around it.<br><br>Berrak: We have always seen the same solids from high school to now. We have always seen the net of a cube, for example, and the net of a pyramid. I think this solid was a little different. Because this is not a solid, we are used to seeing it very often (Task 5).<br>Researcher: Do you know these (composite solids)?<br>Berrak: I do not know any of those.<br>Name: It would be a shame if I could not draw a net of this because it is a solid that we have been making drawings since primary school. Triangular prisms, prisms, and pyramids are the solids that we know and make their unfoldings from primary and secondary school years. |

Table 5 continued

**Drawings:**

| Tuna's drawing for the 1st task | Tuna's drawing for the 2nd task | Tuna's drawing for the 3rd task |
|---|---|---|
|  |  |  |

**Developing strategy:**

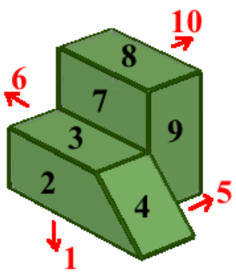
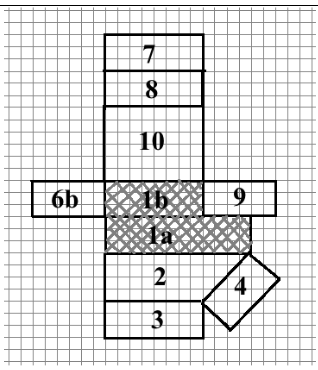
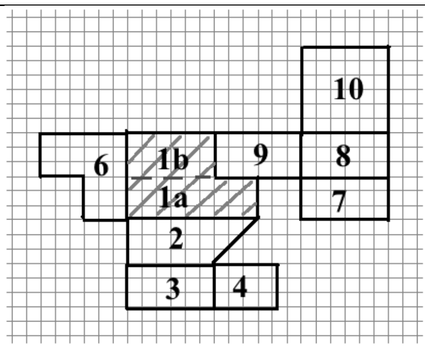
Pera could not develop any strategy to open the surface of the composite solid in the 10th task. Tuna opened the composite solid in the 7th task with a new strategy, which s/he created by coordinating the prototype strategies of opening the side faces around the bases (1a and 1b) of the rectangular prism and trapezoidal prism that form it. On the other hand, Mina developed a more advanced strategy by coordinating a complex strategy used in opening the surface of the rectangular prism with the prototype strategy (opening the side faces around the solid's bottom base) in opening the trapezoid prism. Pera, Tuna and Mina's drawings regarding the 7th task and the interview parts with Pera and Tuna support the study findings.

**Interview parts:**

Pera: I never thought of that. Either the shapes don't fit perfectly anyway. I couldn't open it because nothing was regular. Triangle…Square…Ughh I can't think of it at all. For example, I can't even think about it right now. I can't think of how to open this at all.

Tuna: There was a rectangular prism there (rectangular prism with base 1b). I thought of this (composite solid) as the bases of two joined and adjacent objects... I formed a gap for their intersection (s/he refers to the area where the rectangular prism and the trapezoidal prism overlap)... Then I started to lay the faces down... I opened this object (a vertical trapezoidal prism with a bottom base of 1a). I laid its front face (2), its that face (4) and the top base down (3). Then I connected this face (4) of the front face of this object to this edge (common edge of the front face)... Then I opened this prism (whose bottom base was 1b). First, I opened its side face (9) and its back face (6b) of the prism… I drew this and this one (faces 7 and 8) by laying them towards the back (the face labelled with 10).

**Drawings:**

| The composite solid in the 7 th task | Tuna's drawing | Mina's drawing |
|---|---|---|
|  |  |  |







*3.1.5. Theoretical structure regarding the step "implementing the strategy"*

The research findings have identified the two situations that require careful consideration in order to implement the chosen strategy effectively. *Situation 1:* The planar faces of a solid, which are two-dimensional geometric shapes, are faithfully represented in the solid's net in a way that preserves the edge lengths and angle measurements. That is because opening the solid's surface by rotating its faces along its fold lines is a transformation that conserves the geometric properties of the solid's face shapes. Based on the findings, the transformation of face shapes without distorting their geometric features depends on adhering to the construction rules and procedures (procedural knowledge) and utilizing the shapes' definitions and properties while creating a net drawing of the solid (conceptual knowledge). The transformation also necessitates the proper use of geometric construction tools, such as rulers and protractors, and the properties of the checkered paper. *Situation 2:* Two faces adjacent both in the surface of the solid and that solid's net share similar geometric and spatial relations between their angles and edges. Findings pointed out that considering these relations between adjacent faces ensured the correct positioning of the faces on participants' net drawings —which are geometric shape patterns that form after implementing the chosen strategy.

*3.1.6. Participants' behavioural patterns compatible with the theoretical structure regarding the step "implementing the strategy"*

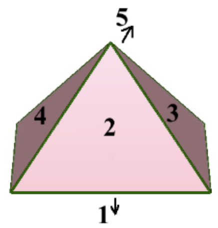
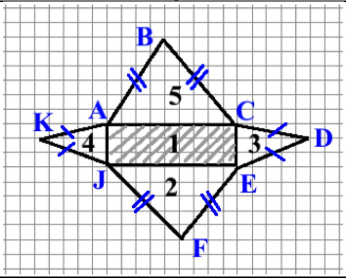
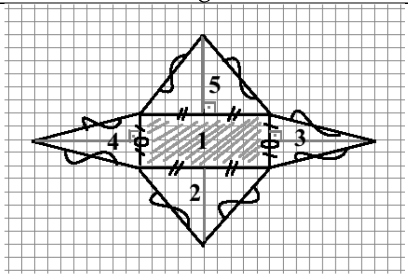
A noteworthy proportion of the participants (15 to 44 participants), successful in devising strategies, did not commit any errors while implementing their chosen strategy in the LCT. In contrast, only drawings of some participants who correctly implemented their chosen strategy (1 to 6 participants) using their conceptual and procedural knowledge were error-free in HCT (see Table 6). A varied number of participants for each LCT (e.g., 14 in the 2nd task and 5 in the 4th task) and HCT task (e.g., 3 participants in the 7th and 8 in the 8th task) were unable to use their conceptual and procedural knowledge at the same time. Hence, they were unsuccessful in accurately projecting the solid's face shapes on their net drawings during the strategy implementation step. Most of the time, even if they had conceptual knowledge, they did not follow the established construction rules and procedures, nor did they use the properties of checkered paper or geometric drawing tools to depict the geometric face shapes of the given solid accurately. A relatively small number of participants encountered difficulties accurately reflecting the spatial and geometric relationships between the shapes present on the solid in their net drawings. As a result, 3 participants made positioning errors in the 2nd task of LCT. Due to the complexity of the tasks, the frequency of positioning errors was higher in the HCT (e.g., 4 participants in the 7th, 3 in the 8th and 6 in the 9th and 10th tasks).

Table 6
*Example behavioural patterns exhibited by the participants in the step of implementing the strategy*

| **Implementing the Strategy** |
|---|
| Hande correctly determined the geometric properties of the face shapes of the pyramid, but she could not correctly reflect these properties in her net drawing. Hande said that the length of [AB] is equal to [BC], though she did not draw these line segments at equal lengths. The situation was similar for other triangles that Hande refers to as isosceles triangles. Dilara made her drawing thinking that the height of an isosceles triangle divides the base into two equal parts, and she accurately reflected all the geometric features of the solid's faces in her net drawing. Hande and Dilara's drawings and excerpts from the interview are examples of the behavioural patterns exhibited by the participants during the strategy implementation phase. |
| **Interview parts:** |
| Hande: The base of this is a rectangle (1); we have two big triangles (2 and 5) and two small triangles (3 and 4). I am not sure, they seem like that. This side ([AB]) and that side ([BC]) have to be equal. When these four (triangles labelled with 2, 3, 4 and 5) are joined, they come together at a point. This side ([KA]) and that side ([AB]) must be equal. I drew it ([KA]) a little smaller. These sides ([KA], [KJ], [CD] and [ED]) and those sides ([AB], [BC], [EF] and [FJ]) must be equal.<br>Researcher: What kind of triangles are these?<br>Hande: … isosceles triangles. This and that isosceles triangle are the same size. These two are also isosceles triangles.<br>Dilara:…I drew it so that the base was rectangular and the side faces were isosceles triangles.<br>Researcher: How did you get them to be isosceles?<br>Dilara: First, I determined the midpoints of these sides. After all, the height needs to divide the base equally, so I drew it accordingly... |

Table 6 continued

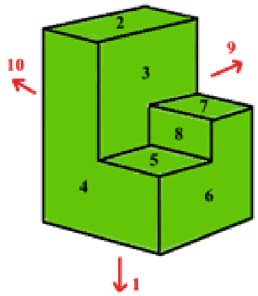
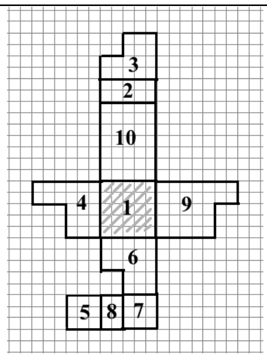
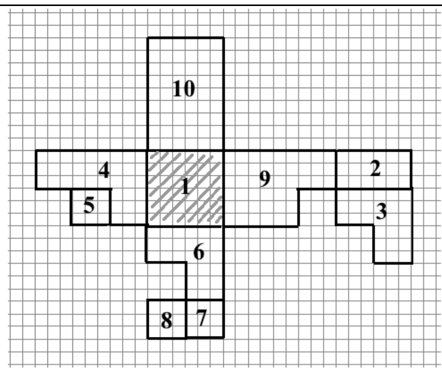
**Drawings:**

| The composite in the 3th task | Hande's drawing for the 3th task | Dilara's drawing for the 3th task |
|---|---|---|
|  |  |  |

**Implementing the Strategy**

Berrin positioned face number 3 incorrectly in her net drawing, which she made for the composite solid. The long edge, which is right of the face she drew, should be on the left, and the short left edge should be on the right. Berrin, who positioned all the other faces correctly in her net drawing, also drew the pairs of edges that will join when the faces are closed of equal length. Sevda made no errors in her net drawing of the composite solid, including positioning errors. Berrin and Sevda's drawings exemplify the findings regarding the behavioural patterns exhibited by the participants in the strategy implementation step.

**Drawings:**

| The composite solid in the 9th task | Berrin's drawing for the 9th task | Sevda's drawing for the 9 th task |
|---|---|---|
|  |  |  |

*3.1.7. Theoretical structure regarding the step "evaluating the solution" (the solid's net drawing)*

When creating a net for a solid, it is essential to meticulously assess each problem-solving step to guarantee that the geometric pattern accurately represents the solid. This scrutiny entails monitoring the problem-solving steps (understanding the configuration or task, developing and applying the strategy) for potential errors and rectifying the detected errors to achieve an exact resolution. The evaluation also involves determining whether the solid's closed surface can be obtained from its net. One can obtain the solid's surface net by rotating the faces around the fold lines, and s/he can get the closed surface of the solid by following the same strategy in reverse order. In attempting to close the solid's net by applying consecutive rotational movements in mind, s/he decides whether the faces' edges that would join were drawn in equal length. The findings revealed that only some participants checked the accuracy of their net drawings by considering various criteria after constructing them. The findings pointed out that most participants conducted a cursory examination of their net drawings, which was insufficient to identify and correct any errors or inaccuracies. For this reason, despite the participants' successful implementation of the chosen strategy, the edges of specific faces failed to converge while attempting to close their net drawings.

### *3.1.8. Participants' behavioural patterns compatible with the theoretical structure regarding the step "evaluating the solution" (the solid's net drawing)*

A varying number of participants throughout the tasks did not control the accuracy of their drawings or conducted cursory checks that were insufficient in detecting their errors. Fewer participants in the HCT reflected all faces of solids into their net drawings compared to the LCT. In the initial and last tasks of the LCT, respectively, only one participant and four participants did not reflect one of the solid's faces in their net drawings. However, the number of participants who committed this error was higher in the HCT (e.g., 6 participants in the 6th, 6 in the 9th and 3 in the 10th task). Although the participants' net drawings appeared to be accurate, specific edge pairs that should have been of equal lengths were drawn in different lengths (e.g., 1 participant in the 1st, 8 in the 4th and 6 in the 5th of the LCT and 3 participants in the 8th, 3 in the 9th and 1 in the 10th of the HCT). The differences in these edge lengths, which should have joined when the faces folded, hindered obtaining the closed surface of the solid from the participants' net drawings.

Table 7

*Example behavioural patterns exhibited by the participants in the step of evaluating the solid's net*

**Evaluating the solid's net:**

Naz drew all the faces of the triangular prism according to their geometric properties. However, she drew most of the edge pairs in different lengths instead of equal lengths. Dilara thought of the top and bottom faces of the solid as a right triangle. After assigning specific length values to the two sides of one of these right triangles, she calculated the third side from the Pythagorean theorem. Dilara drew her net accurately using the length values she assigned the triangle's edges. The interview made with Naz and the net drawings of Naz and Dilara exemplify the findings regarding the evaluation step of the problem-solving process.

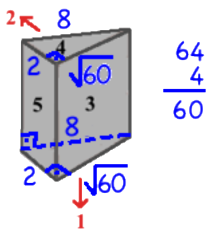
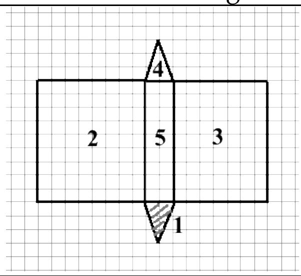
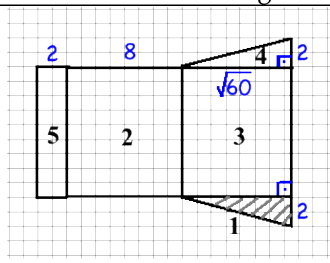
**Interview part:**

Naz: I had a bit of a problem with the size here because I couldn't accurately draw the side lengths of the triangle.

Researcher: How do you think it should have been?

Naz: I do not think the exact sizes of these sides. The triangle should have been higher in height, or the rectangles should have been smaller in width.

**Drawings:**

| The solid in the 2nd task | Naz's drawing | Dilara's drawing |
|---|---|---|
|  |  |  |







**Evaluating the solid's net:**

Suna opened the composite solid's surface in the 7th task by coordinating the trapezoid and square prism's prototype unfolding strategies. Suna made her drawing by considering the edge lengths that would join when her net drawing is folded. However, she realized that she had forgotten to reflect face number 7 but did not realize that she had not reflected face number 5 of the composite solid in her net drawing. Nevin reflected all the faces of the composite solid in her net drawing and drew the edge pairs in equal lengths that would join when her net was closed.

**Interview part:**

Suna: When I close it, this forms the front face (2). When I fold this, it forms the top base (3). This side and that side meet (sides n). This side and that side meet (sides m). The lengths of these sides are equal (sides m)...I drew this face (4) longer than that face (6a). I drew this face (3) shorter than that face (1a)...(She began to explain how she closed the faces square prism). Likewise, I folded that face (9). These are two equal rectangles this and that (6b and 9). However, when I closed, I could not form this face (7). That remained as a gap...This face forms the back face (10). This one is the top base (8).

Table 7 continued

<table>
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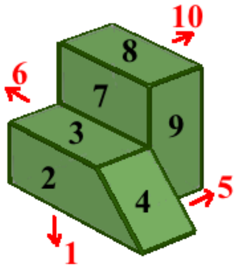
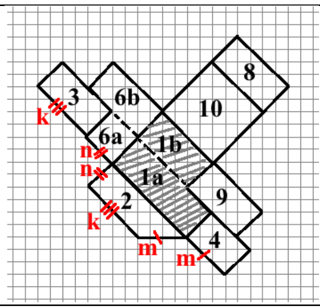
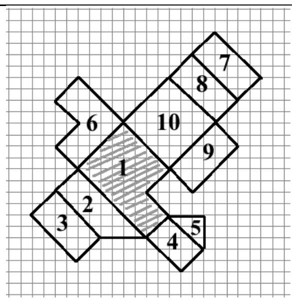
<tr><td>
</td><td>
</td><td>
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### 3.2. Participants’ Development Stages of Spatial Visualisation Generalisation Processes

The verified hypotheses showed that participants’ spatial visualisation generalisation processes went through certain developmental stages: novice, competent, and expert. Participants’ behavioural patterns regarding their problem-solving schemas’ development characterised and helped define those stages. The stages, compatible with the novice-expert theoretical framework, were defined according to the participants’ drawings with or without errors, whether any challenge was experienced in the problem-solving steps and at what problem-solving step they experienced challenges. The findings from the tested and verified hypotheses enabled the definition of the novice, competent and expert stages characterised by participant behaviour patterns, drawings and interview parts. It was given place to the numerical data addressing how many participants were at the novice, competent, or expert stage at each LCT and HCT task.

#### *3.2.1. Development stages of spatial visualisation problem-solving schemas for LCT*

In the Novice Stage (see Table 8), participants encountered difficulty comprehending the configuration of the given basic solid. Specifically, they lacked understanding regarding how to open the solid, as highlighted in the task. For instance, neither they take the base of the solid as the reference point to unfold it nor make their drawings in a manner that would make its inner surface visible. Moving on to the Competence Stage (see Table 9), the participants better understood the solid’s configuration but still encountered challenges in reasoning about the geometric properties of some faces. Despite comprehending the task requirements and devising a suitable strategy to unfold the solid, they struggled to construct the nets accurately and did not evaluate their drawings’ precision; that is, their net drawings were not free from errors. In the Expert Stage (see Table 10), participants consciously managed their problem-solving process without encountering obstacles in any problem-solving steps. They successfully made their net drawings with accuracy, while some even demonstrated their proficiency by drawing a correct net of the solid using a more complex strategy. However, irrespective of the developmental stage, prototype opening strategies were the most preferred approach for opening the surface of basic solids.

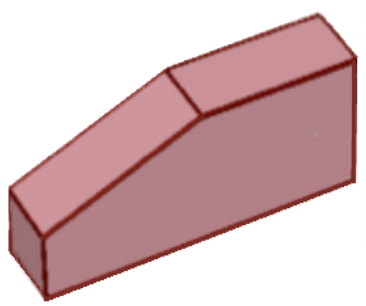
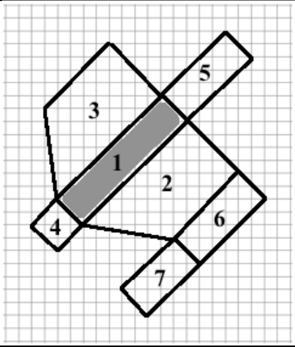
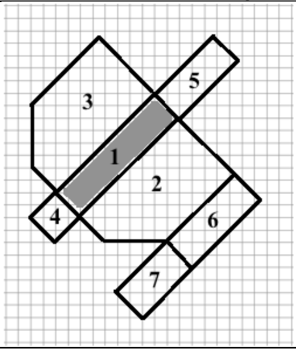
Table 8

*Sample problem-solving behavioural patterns exhibited by participants in the novice stage for LCT*

**Participants in Novice Stage**

**The participant's problem-solving behavioural patterns:** Gülden made the internal representation of the pentagonal prism as a right trapezoidal prism.

**Drawings:**

| The solid in the 5th task | Gulden's drawing | Accurate drawing |
|---|---|---|
|  |  |  |

**The interview part, made with Gülden:**

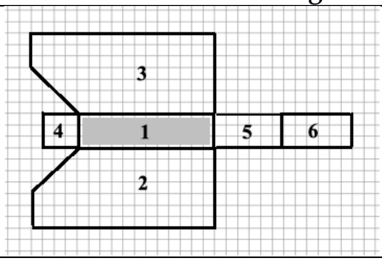
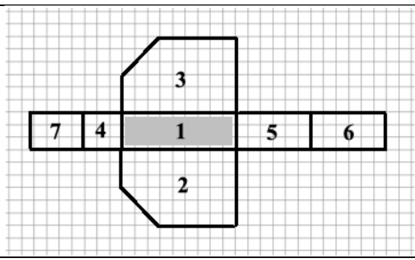
Gülden: The bottom of it is a rectangle (1), namely, the place where the solid is put on is a rectangle...the left side of the solid is a square. I think this side is the left side, and it is a small square (4). According to me, these sides are front and back and are trapezoidal. I can say right trapezoid, yes, right trapezoid (2 and 3). The other side, namely the right side of it, is a rectangle (5).

Researcher: What shapes are its front and back faces?

Gülden: They are trapezoids, right trapezoids.

**The participant's problem-solving behavioural patterns:** Berra unfolded the pentagonal prism by taking its upper base as the reference face instead of its bottom base. When Berra unfolded the solid, its external surface was visible, not its internal surface.

**Drawings:**

| **Berra's drawing** | **Accurate drawing** |
|---|---|
|  |  |

**The interview part, made with Berra:**

Berra: First, I lifted up this face (2) and the face opposite of it (3). Then I lifted up that square face (4) and the rectangular face (5) opposite of it. Lastly, I added the remaining face (6) to this rectangular face.

Researcher: Which face did you keep fixed?

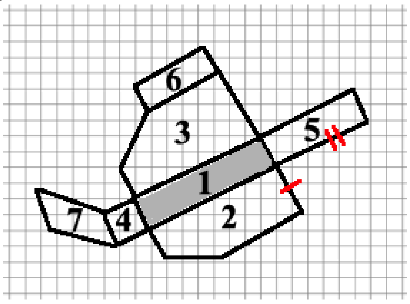
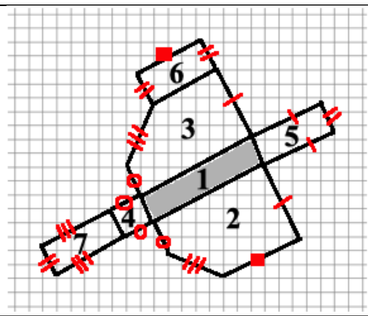
Berra: This one (1).

Table 9
*Sample problem-solving behavioural patterns exhibited by participants in the competent stage for LCT*

**Participants in Competent Stage**

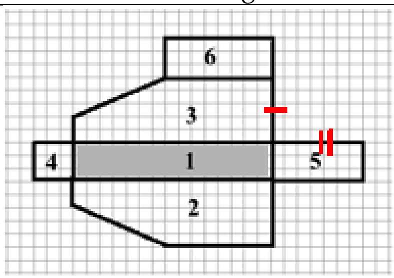
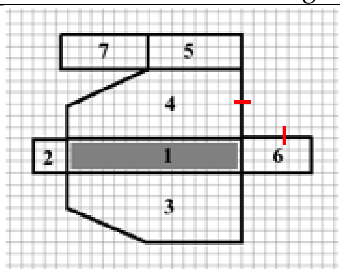
**The participant's problem-solving behavioural patterns:** Derin could not deduce that the solid's oblique face (7) must be rectangular and did not check that the edge pairs in his/her drawing that would join when she folded her net were equal in length. He/she did not make A1 and A3 errors in his/her drawing.

**Drawings:**

| Derin's drawing | Accurate drawing |
| --- | --- |
|  |  |

**The participant's problem-solving behavioural patterns:** Tülin did not check whether the number of faces of the solid and faces of her net drawing were equal so he/she did not draw the face labelled with 7. He/she also did not check the edge pairs in his/her drawing that would join when she fold her net were in equal length. He/she made one A1 error, but she drew the faces by considering their geometrical properties and positioned them correctly.

**Drawings:**

| Tulin's drawing | Accurate drawing |
| --- | --- |
|  |  |

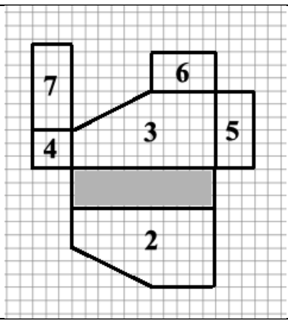
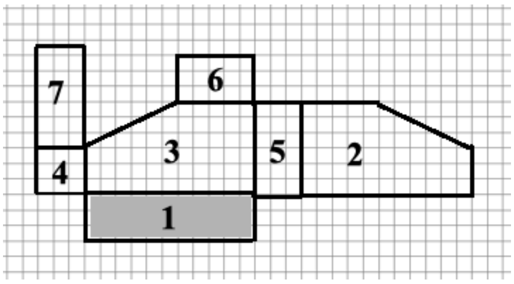
**The interview part, made with Tulin:**
Tulin: I take a rectangular (1) for its base… I drew its front face, pentagonal (2)….Then I opened its left and right faces (4 and 5)….and then I drew the other pentagonal face (3), opposite to this face (2), and then I added this rectangular face (6) to this face (3).
Reseacher: Did you take account anything while you were drawing?
Tulin: For example, I tried to make its front and back faces in equal size…
Reseacher: What did you do to close your net?
Tulin: For example, I drew this face (5) elongated so that it would be the same size as this one (3).
Researcher: Did you take into account their lengths? Were they equal lengths?
Tulin: I tried to take into account, but I think I could not draw them in same lengths.

Table 10
*Sample problem-solving behavioural patterns exhibited by the participant in the expert stage for LCT*

**Participants in Expert Stage**

**The participant's problem-solving behavioural patterns:** Omur developed a different strategy to unfold the solid compared to the other participants. He/she implemented his/her strategy correctly. He/she also took into account edge pairs that would join when he/she folds his/her net drawing.

**Drawings:**

| Omur's accurate drawing | Accurate drawing after implemented a complex strategy |
|---|---|
|  |  |





**The interview part, made with Omur:**
Omur: This (1) is the bottom of the solid. First, I unfolded this front face (2). I took this pentagonal face as the front face. I unfolded the back face (3) that was opposite position to the front face. I unfolded this small square face (4) and this right face (5) to the back face. Similarly, I added this face (6) to the back face (3), and this face (7) to this small square. I connected all these faces (3, 4, 5, 6 and) to the back face.
Researcher: Did you consider anything while drawing?
Omur: For example, I drew this edge and this edge in equal length. I drew them 3 units long.

### *3.2.2. Development stages of spatial visualisation problem-solving schemas for HCT*

Participants in the novice stage (See Table 11) could not accurately make the mental representation of a composite solid. They failed to recognize that composite solids possess distinct geometric properties different from their constituent-basic solids. In creating a net of the composite solid, most drew the nets of its constituent-basic solids separately and attempted to join them via a common edge. In the competence stage (See Table 12), participants explored the geometrical properties of composite solids that differed from their constituent-basic solids' properties. They developed strategies for unfolding composite solids, coordinating the prototype unfolding strategies of basic solids. Despite this progress, however, the participants still possessed certain misconceptions or misunderstandings regarding the properties of the composite solids and their nets. As an example of this case, net drawings of participants, which resulted in forming two closed surfaces instead of one when folded, can be given. Advanced problem-solving schemes define the expert stage (See Table 13), in which participants drew the net of the composite solid without any error. This situation implied that the participants in this stage checked the accuracy of their solutions. Some participants also developed more complex unfolding strategies by choosing non-prototypical strategies of the basic solids that form the composite solid.

Table 11

*Sample problem-solving behavioural patterns exhibited by participants in the novice stage for HCT*

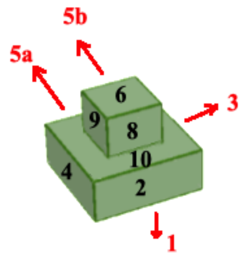
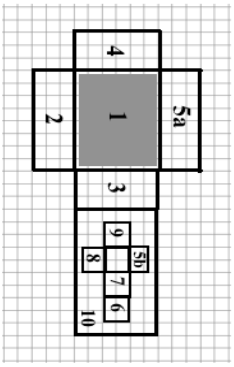
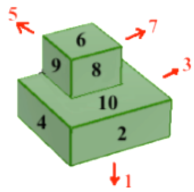
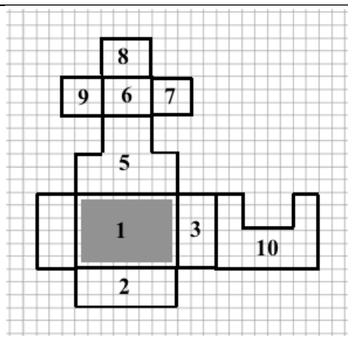
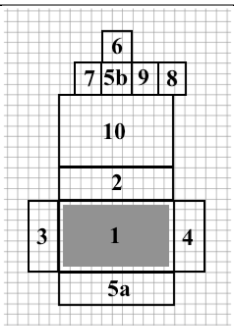
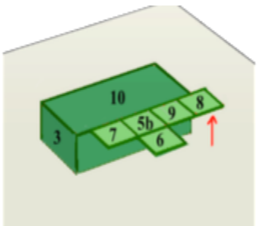
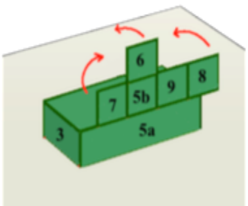
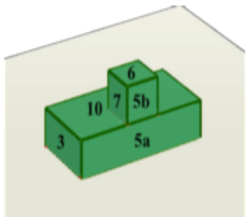
<table>
<tr><th colspan="4">Participants in Novice Stage</th></tr>
<tr><td colspan="4">The participant's problem-solving behavioural patterns: Ebrar and Ayfer conjectured that the composite solid is formed from a square prism and a cube that was placed on the middle of the top-face of this square prism, so Ebrar did not draw its net but Ayfer developed a wrong strategy to unfold it.</td></tr>
<tr><td colspan="4">Drawings:</td></tr>
<tr><td>The mental representation of Ebrar and Ayfer for the solid in the 6th task</td><td>Ayfer's drawing</td><td>The presented composite solid in the 6th task</td><td>Accurate drawing</td></tr>
<tr><td>
</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td colspan="4">The interview part, made with Ayfer:<br>Ayfer: I unfolded this prism like this (the faces labelled with 1, 2, 3, 4, 5a and 10) ... I unfolded this cube (5b, 6, 7, 8 and 9) to here. I opened the cube to its the top-face (10) ... I tried to keep the cube on the top.<br>Researcher: Did you unfold this cube to the top-face of that square prism.<br>Ayfer: Yes.</td></tr>
<tr><td colspan="4">The participant's problem-solving behavioural patterns: Although Tugce made the internal representation of the composite solid correctly, he/she did not have sufficient knowledge about the surface and the faces of a composite solid or properties of a net of a solid. Threfore, he/she developed a wrong strategy to unfold this composite solid.</td></tr>
<tr><td colspan="4">Drawings:</td></tr>
<tr><td>Tugce's drawing</td><td>Tugce's folding process</td><td>Tuğçe's folding process</td><td>The internal image of Tugce for the solid in the 6th task</td></tr>
<tr><td>
</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td colspan="4">The interview part, made with Tugce:<br>Tugce: It looks like there's a cube placed on top of its rectangular prism.<br>Researcher: Where does the cube stand? Is it in the middle of the upper face of the prism that was below?<br>Tugce: It is both in middle and not in the middle. Actually, it is not in the middle. How can I tell you? Their back faces are adjacent....<br>Reseacher: How did you unfold?<br>Tugce: First of all, I opened the side faces around the base (2, 3, 4 and 5a). After that, I opened that top face (the face labelled with 10 that was drawn like a rectangular) to the backwards, and then I opened the cube (6, 7, 8, 9 and 5b).</td></tr>
</table>

Table 11 continued

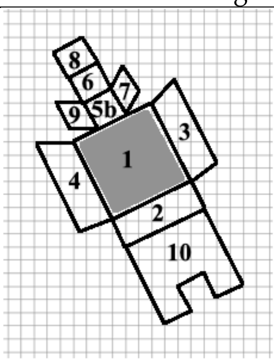
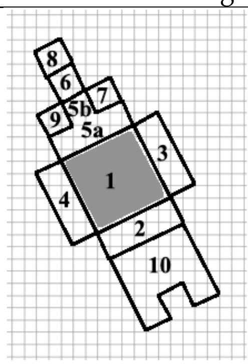
Researcher: I don't quite understand how you opened the cube there (At the beginning, the researcher did not understand the strategy developed by Tugce to unfold the solid)….
Researcher: How do you close your net?
Tugce: Firstly, I closed the side faces around the base (2, 3, 4 and 5a). When I folded this face (10), it stood upper the bottom face (1). When I folded this face (10), these faces (5b, 6, 7, 8 and 9) carried out with it. I tried to close these shapes (5b, 6, 7, 8 and 9) so that they formed a cube on this large rectangular (10). I folded these squares (5b, 6, 7, 8 and 9) backwards, not forwards….
Researcher: How was the position of the solid while you were opening? How was the appearance of it?
Tugce: How can I tell you?... I opened it by thinking as if its back was looking at me.

Table 12
*Sample problem-solving behavioural patterns exhibited by participants in the competent stage for HCT*

**Participants in Competent Stage**

**The participant's problem-solving behavioural patterns:** Nermin made the solid's internal representation in his/her mind correctly and were able to develop a strategy to unfold the solid. However he/she forgot to draw the sub-part (5a) of the T-shape face. This situation triggered his/her errors. He/she did not reason about geometrical properties of some faces of it so there were deformation of geometric properties of some drawn faces. He/she perceived square and rectangular faces of the solid as parallelograms and trapezoids.

**Drawings:**

| Nermin's drawing | Accurate drawing |
|---|---|
|  |  |

**The participant's problem-solving behavioural patterns:** Hilal were able to coordinate unfolding strategies of the cube and the prism and developed a correct strategy. However, he/she drew an extra face on his/her net drawing and the U-shaped face as a rectangle. When he/she folded his/her drawing, two closed surface is obtained, though he/she should have obtained one closed surface.

**Drawings:**

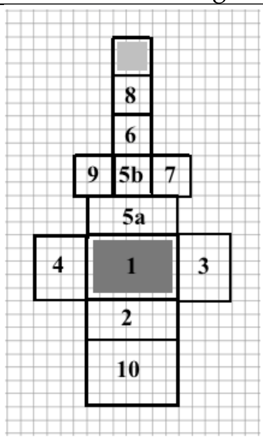
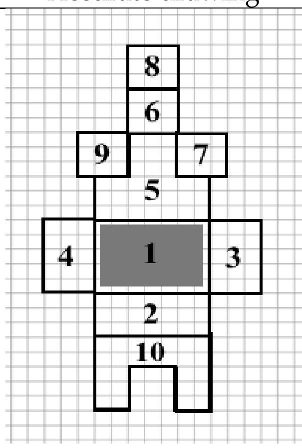
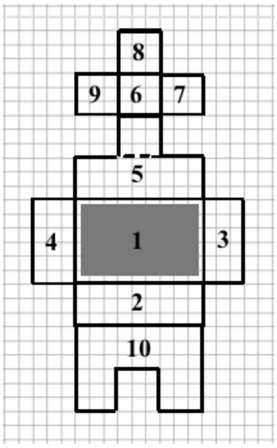
| Hilal's drawing | Accurate drawing |
|---|---|
|  |  |

Table 12 continued

| **The interview part, made with Hilal:** |
|---|
| Hilal: The cube is above, and the prisma is below. Was that what type of prism? Is it a rectangular prism? or Is it a square prism? The back of the prism and the back of the cube are in same alignment.<br>Hilal: First, I unfolded prism… I drew this rectangular (1) to represent the bottom of the solid. Then, I drew this rectangulars to represent its left and right faces (3 and 4). I drew this to represent its front face. This is top-face (10) of the prism. When I fold this face (10), it stands above the bottom face (1). This (5a) is the opposite face of the front face (2). These squares (5b, 6, 7, 8, 9 and light grey faces) belong to the cube. This (5b) is back face of the cube. These two represent these two (7 and 9), and this square is the top face of the solid (6). This one (the light grey face) is the bottom face of the cube. When I fold the cube, this face (light grey face) stays at the bottom of the cube. |

Table 13

*Sample problem-solving behavioural patterns exhibited by the participant in the expert stage for HCT*

| **Participants in Expert Stage** | |
|---|---|
| **The participant's problem-solving behavioural patterns:** Sena drew a net of the composite solid correctly. He/she unfolded the surface of the composite solid by coordinating the strategy of the cube and prism. He/she drew the edge pairs in equal length that would join when he/she folded his/her net drawing. | |
| **Drawing and Interview Parts:** | |
| Sena's Correct Drawing | Interview part, made with Sena |
|   | Sena: I thought that a cube was placed on a rectangular prism, and I tried to unfold according to it.<br>Researcher: Can you describe the solid a little bit?<br>Sena: As far as I understand, the back faces (5a and 5b) are adjacent. We must take the back faces as a whole (the invisible T-shaped face)<br>Researcher: How did you unfold the faces of the solid? In what order?<br>Sena: I lied these side faces (2, 3, 4 and 5) to the ground . I drew the upper base of the rectangular prism (10) by subtracting a space equal to the cube. I subtracted a square, equal to the bottom base of the cube, from this rectangular (10) (he/she is describing the U-shaped face). The plus-shaped part (6, 7, 8, 9 and the top part of the face 5) of my drawing represents the cube's net. I unfolded the cube, here. I didn't put the bottom base of the cube. That's why I had removed this part. I made sure that there was a gap between the prism and cube. I opened the side faces (7, 8 and 9) of the cube upwards, that is, to the upper base (6) of the cube.<br>Researcher: Did you pay attention the lengths of the edges in your drawing?<br>Sena: For example, I take 3 units long these edges. If I lift up these faces, they will join with each other. |

### *3.2.3. The number of participants in different developmental problem-solving stages for LCT and HCT*

Table14 provides information abut two situations: (a) how many participants, grouped as low, medium, and high performers based on their PSVT scores, demonstrated that their problem-solving behavior patterns were at the novice, competent, or expert stages? (b) what is the distribution of participants exhibiting novice, competent and expert problem-solving patterns in each task?

Table 14
*Distribution of low, medium and high PSVT performer participants onto the novice, competent and expert stages in each task of LCT and HCT*

| *Development Stages* | | | *LCT* | | | | | *HCT* | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Novice | | | | | | | | | | |
| Low | 0 | 2 | 3 | 2 | 9 | 14 | 13 | 12 | 10 | 14 |
| Medium | 0 | 2 | 0 | 2 | 6 | 11 | 14 | 7 | 5 | 5 |
| High | 0 | 0 | 0 | 0 | 1 | 2 | 2 | 1 | 2 | 3 |
| Total | 0 | 4 | 3 | 4 | 16 | 27 | 30 | 20 | 17 | 22 |
| Compotent | | | | | | | | | | |
| Low | 3 | 10 | 6 | 11 | 5 | 1 | 2 | 2 | 4 | 1 |
| Medium | 0 | 11 | 10 | 8 | 2 | 7 | 4 | 9 | 11 | 12 |
| High | 0 | 4 | 3 | 2 | 3 | 7 | 7 | 7 | 6 | 6 |
| Total | 3 | 25 | 19 | 20 | 10 | 15 | 13 | 19 | 21 | 19 |
| Expert | | | | | | | | | | |
| Low | 12 | 3 | 4 | 2 | 1 | 0 | 0 | 1 | 1 | 0 |
| Medium | 18 | 5 | 8 | 8 | 10 | 0 | 0 | 2 | 2 | 1 |
| High | 11 | 7 | 10 | 9 | 7 | 2 | 1 | 2 | 3 | 2 |
| Total | 41 | 15 | 22 | 20 | 18 | 2 | 1 | 5 | 6 | 3 |

Most low-performer participants exhibited behavioural patterns suggesting that they were competent in tasks 2, 3, and 4, while high-performers' behavioural patterns indicated that they were at the expert stage. Meanwhile, many medium performers demonstrated behavioral patterns signifying their attainment of competence or expertise in those three tasks. The great proportion of low-performers displayed characteristic problem-solving behaviours akin to those of novices during task 5 and the HCT tasks. Many medium performers' problem-solving tendencies were at the novice stage in Tasks 6 and 7, while they were at either novice or competent behaviour in the remaining HCT tasks. A significant number of high-performers' problem-solving schemas were at expert stage on taks 5, and at competent stage in HCTs. The participants with expert behavioral patterns in the 6th and 7th tasks were exclusively among high-performers, while they were among both medium and high performers in the 10th task. Although participants displaying expert problem-solving behavior in Tasks 8 and 9 were among medium or high performers predominantly, it was noted that one low-performer also exhibited expert problem-solving behavior in these tasks.

In the initial four tasks of the LCT, there were only a small number of participants exhibiting characteristics of the novice stage, but this number notably increased in the 5th task (see Table 14). Most participants showed expert-stage characteristics in the first task of the LCT, with a few exceptions. However, in the other tasks of the LCT, many participants portrayed characteristics of both the competent and expert stages. In contrast to these findings, in each of the tasks in the HCT, the majority of participants exhibited characteristics of the novice or competent stage. There were relatively few participants who demonstrated expertise in the HCT tasks, although the numbers varied from task to task.

## 4. Discussion

The research findings demonstrated that pre-service mathematics teachers' processes of reaching spatial generalisations of drawing the surface nets of solids were compatible with the two theoretical frameworks: Polya's problem-solving algorithm and novice-to-expert problem-solving schemas. The problem-solving processes of the participants involved a series of steps, including the creation of a mental representation of the problem situation (comprehending the configuration and the requirements of the task), devising an appropriate strategy to unfold the surface of the solid, implementing the strategy enabling to draw one of the nets of the solid, and finally evaluating the accuracy of the net drawing representing the opened surface of the solid. The present research establishes that learners' attainment of spatial generalisations undergoes a developmental process characterized by three key stages —novice, competent, and expert— which can be accurately predicted based on their performance in spatial visualization problem-solving tasks. A comprehensive grasp of the fundamental mechanisms underlying participant performance variations is imperative for augmenting academic achievement. Therefore, researchers are increasingly interested in analyzing differences in learners' behaviours exhibited in educational assessments rather than simply assessing outcomes. Such analyses offer valuable information about the differences in the cognitive structures of successful and unsuccessful learners and assist in designing suitable educational interventions (Stadler et al., 2019).

When faced with a problem for the first time, individuals must seek to fully comprehend the nature of the issue before attempting to devise a solution (Akin, 2022; Ozturk et al., 2020). This initial step holds a pivotal role in setting the course for any subsequent actions that may be undertaken. In situations where a problem is presented in written form, the ability to connect and reason with relevant mathematical knowledge and skills is of utmost importance in achieving a comprehensive understanding of the issue at hand. Whereas, if given a problem involving a spatial configuration, as in this study, cognitive processes such as grasping and visualizing spatial or geometric shapes need to be activated in order to understand it (Akin, 2022; Unlu & Ertekin, 2017). The initial step for comprehending the problem situation is to transform the information presented through external representations into internal representations, which can be achieved by processing that information via perceptual mechanisms (Fischer et al., 2012; Zhang, 1997). This study found that expert and most competent participants were able to visualize the solid correctly from its 2D representation, knew geometric concepts related to the task (e.g. solid and its components, its surface and nets), understood how to open the surface of the solid, and developed insight about how to draw the solid's net by associating the given information in the task with the represented configuration of the solid. The initial stage of problem-solving, which entails capturing and representing pertinent information about the situation, is a critical determinant of the ultimate success or failure of the solution process (Pribyl & Bodner, 1987). Unlike the participants in competent and expert stages, the study revealed that novice participants failed to create an appropriate internal representation of the given spatial task, which led them to make radical and logical errors, preventing them from accessing an accurate solution. According to Lewis (1989), most problem-solving errors occur during the initial step, where the problem is often misrepresented in the mind rather than during the strategy execution. It is important to note that accurate mental representation of a problem is crucial for correct decision-making and implementation of operations required for a solution (Schnotz et al., 2010; Zhang, 1997).

The strategy development involves learners' utilization of generalised actions that have been employed previously to solve similar problems. Learners either apply these generalised actions directly to newly encountered problems or modify and combine them to achieve the desired outcome (Rott et al., 2021). This approach allows learners to efficiently and effectively solve complex problems by calling the appropriate strategy from their repository of experiences. In this study, the participants who demonstrated success in solving spatial tasks were observed to utilize prototype strategies for basic solids, while they opted for strategies developed through the modification and amalgamation of those prototype strategies in the case of composite solids. These findings are in line with the study of Rott et al. (2021). Apart from this, this study has also manifested that the novice participants' inadequate conceptual knowledge led to their inability to accurately represent the problem, thus impeding them from developing a viable strategy. In contrast, the strategies implemented by expert or skilled competent problem-solvers surpassed the mere utilization of prototypes or their combination and showcased their potential to generate more sophisticated strategies. Moreover, most participants demonstrated competent or expert problem-solving behaviour patterns in cases where prototype strategies were adequate. On the other hand, in more complex spatial tasks in which combination and coordination of prototype strategies were necessary or a new strategy was preferred, most participants exhibited novice problem-solving behaviour patterns. The study findings regarding the problem-solving strategies of novices are consistent with the assertion made by the research of Persky and Robinson (2017), where they posited that when learners encounter difficulties in comprehending a problem, they develop a flawed mental representation of it, leading to the application of inappropriate problem-solving strategies that adversely impact the process. In agreement with this study's findings concerning complex tasks, Price et al. (2022) state that experts relying on their existing knowledge base tend to develop more conscious and reflective strategies, unlike novices, even when confronted with unfamiliar problems or complex tasks.

Stadler et al. (2019) have identified a distinction between learners who can select the correct problem-solving strategy but struggle with its error-free implementation, and learners who can successfully apply the selected strategy without any errors. The study noticed that participants who applied their chosen strategy successfully could effectively utilize and coordinate their conceptual and procedural knowledge. In contrast, those who could not apply it faced difficulties reflecting the faces' geometric features on their net drawings. In addition, participants who paid meticulous attention to the geometric and spatial relationships between the face shapes, both in the solid form and when represented as a net, could apply their developed strategy without error. On the other hand, those who did not gain expertise in the tasks made positioning errors. The first finding is compatible with the study of Braithwaite and Sprague (2021), drawing attention to the importance of conceptual and procedural knowledge in problem-solving. They emphasized that learners prioritize procedures over concepts in most contexts in strategy implementation. However, in situations where procedural knowledge alone is not adequate to solve the problem, they use conceptual knowledge as a supplement. The second finding aligns with the study conducted by Sung and Park (2012), wherein they highlighted the significance of spatial abilities, particularly the capacity to manipulate 2D or 3D figures mentally, in the efficient execution of solution strategies.

Mathematics education research has frequently reported that learners overlook the crucial step of evaluating their work during or after solving problems despite the widespread recognition of this process's significance. On the other hand, evidence suggests a strong positive correlation between successful problem-solving and carefully evaluating one's work (Kontorovich, 2019). In this study, consistent with the previous research conducted in mathematics education, it has been observed that only a particular part of the participants performed a comprehensive evaluation of their net drawings by scrutinizing multiple criteria after their creation. On the other hand, most participants conducted a superficial examination, not allowing them to find errors in their net drawings or did not make any evaluations. Furthermore, more participants fulfilled the requirements of all problem-solving steps, including checking their solutions' accuracy in low-complexity tasks. However, in the case of high-complexity tasks, a relatively low number of participants checked the accuracy of their solutions. This finding is affirmed by Kontorovich's (2019) study, which highlighted that checking a solution in certain problem situations can be relatively less cognitively demanding than in others. This situation brings attention to the interplay between the availability of checking strategies and the decision to execute them. From this standpoint, it is plausible to state that in low-complexity tasks, more study participants evaluated their solutions, as the associated controls required less cognitive effort. However, only some participants could execute the checks for high-complexity tasks due to the extreme cognitive demands they entail.

The study found the number of participants with a low problem-solving schema who could handle low-complexity spatial visualization tasks was more numerous than those with a high-order problem-solving schema who could handle high-complexity tasks. According to the schema theory, skilled performance is achieved by creating increasingly complex schemas in which the elements of lower-level schemas combine into higher-level schemas (Sweller et al., 1998). The inadequate knowledge and skills, as well as the cognitive demands of the tasks, caused the participants to exhibit behavioural patterns, indicating that they were at different developmental stages in different spatial visualization tasks. By creating comprehensive and well-structured schemas, learners can perform better on more complex tasks. Participants' errors in specific solution steps of spatial visualisation tasks and the proximity of their solutions to an accurate solution helped to define the spatial visualisation developmental stages. This finding is in line with the study of Nokes et al. (2010), reporting that learners' domain knowledge and skills impact each step of the problem-solving process, from making a mental representation of the problem to evaluating the solution. Because of that, experts and novices can make a problem's mental representation very differently, even when looking at the same stimulus. Likewise, the study participants in the novice stage struggled to represent spatial visualisation tasks accurately or develop effective strategies, contrary to participants in other stages. However, participants at the competent and expert stages tended to use prototype strategies derived from their prior experiences or strategies they developed by combining and coordinating these strategies. These findings can be explained by the studies of Nokes et al. (2010) and Pretz et al. (2003). Learners make connections between their prior knowledge and the related task when choosing or implementing strategies (Nokes et al., 2010). During their problem-solving process, learners retrieve an analogical solution from their memories based on clues and patterns, as described by Pretz et al. (2003). If the analogical solution aligns with the encountered task, or if they have the necessary knowledge and skills to bridge any gaps between the solution and the task at hand, they can solve it correctly. However, they cannot reach a correct solution if the analogical solution does not align with that task. This study concluded that participants who completed spatial visualization tasks without errors evaluated their solutions. This finding is consistent with that of Nokes et al. (2010), indicating that experts are more adept at recognizing and rectifying errors than novices. Apart from these, this study also implies that there may be significant relationships between participants' PSVT performances (low, medium, and high) and the development of their problem-solving schemas, categorized as novice, competent, or expert stages. This finding may be the focus of future quantitative studies.

Gaining expertise requires experiencing several stages, and it is essential to note that a learner cannot transition directly from novice to expert. Indeed, a learner must experience intermediate stages before achieving expert status. It is also worth noting that during this process, an individual may simultaneously exhibit characteristics of two stages (novice/expert and intermediate) (Persky & Robinson, 2017). In parallel with those, this study identified that participants went through three stages —novice, competent and expert— in reaching spatial visualization generalisations. The participants in the novice stage lacked sufficient conceptual and procedural knowledge and skills to solve the task at hand. In addition, they could not organize their knowledge and skills in a manner that allowed them to accurately represent the problem in their mind or develop a correct strategy to solve it. Because they managed the problem-solving process unconsciously, they made mistakes in understanding the task and developing strategies, which radically hindered them from reaching an accurate solution. In contrast, participants in the competent stage tried to manage their problem-solving procedures consciously by employing their existing conceptual knowledge and skills. However, despite their efforts, they could not attain an error-free solution. The amount and quality of conceptual and procedural knowledge and skills of expert stage participants, as well as their ability to manage the problem-solving process in a highly conscious manner, enabled them to solve the tasks they encountered without error. The characteristics of spatial visualisation generalisation stages in this study are compatible with the abstraction stages of Cifarelli's (1988) problem-solving process. According to Cifarelli (1988), mathematical problem solutions go through four stages of abstraction. In the first stage (recognition), learners realize that the new problem they face is similar to the ones they have solved before. However, they cannot predict the difficulties that may arise due to the unique features of the new problem situation (Goodson-Espy, 1998). The second one (re-representation) is the stage where learners can distinguish the similarities and differences between the problems they already know the solution to and the new problem situation, but they cannot think about potential strategies and their solutions. In the third stage (structural abstraction), the learner can determine a solution method based on the distinctive features of the new problem and realize this consciously by adapting the strategies they have used before and considering potential solutions (Goodson-Espy, 1998). The last one is referenced as structural awareness whereby conscious solutions are derived through the flexible utilization of mathematical concepts and skills.

## 5. Conclusions

Matching spatial ability with intelligence (Galton, 1883), expressing it as an "innate ability to visualise that a person has (Sorby, 1999)" and focusing on the connections between performance on spatial ability tests and age, gender, and occupation types has led to the idea that spatial visualisation ability is a talent that not everyone has at the same level and one that can be inherited from genes. Moreover, the automaticity and practicality of experts' spatial visualisation performances pose a challenge in comprehending the sequential problem-solving steps passed through while they perform spatial visualisation tasks. Likewise, the stages of cognitive development that individuals undergo before achieving expertise in spatial visualization tasks are often overlooked. Based on theory-testing, this study achieved remarkable findings revealing that pre-service mathematics teachers' processes of reaching spatial visualisation generalisations were compatible with the two theories of Polya's problem-solving steps and novice-to-expertise problem-solving schemas. The participants' process for solving spatial visualisation tasks involved a structured approach comprising four sequential steps. These steps encompassed understanding the task, formulating a strategy, executing the strategy, and evaluating the solution. The process of achieving spatial generalisations occurred in three sequential stages: novice, competent, and expert. The expert stage is characterised by participants' conscious problem-solving management, ultimately leading to an accurate solution. Even if the problem-solving process tries to be managed consciously, a minor mistake in any step can lead to failure. This outcome defines the competent stage. The novice stage involves an unconscious problem-solving process, which leads to mistakes in understanding the problem and developing a strategy, radically affecting the solution. The ease of constructing low-order schemas led to more participants in the LCT exhibiting expert-stage behavioural patterns. Due to the excessive effort required for constructing high-order schemas, fewer participants demonstrated expert-stage behavioural patterns in the HCT.

Understanding how spatial generalisations are reached can provide valuable insights into developing practical and functional approaches for teaching learners to solve spatial visualisation problems and improve their ability to generalise solutions by identifying similarities, differences, and structural connections (Mulligan et al., 2018). Comprehending the progression of spatial visualisation ability may also be instrumental in enabling educators and researchers to devise suitable learning experiences for students at varying levels, thus facilitating the development of their spatial reasoning skills (Harris, 2021). In addition to its potential, this study has certain limitations in various aspects including the sample size, the used spatial visualisation tasks, distribution rates of participants according to gender and not focusing on possible connections between participants' spatial test performances and spatial generalisation processes. To strengthen the quantitative aspect of the study by preserving its qualitative aspect, it is recommended to conduct focus group discussions with participants who share similar characteristics (e.g., reveal similar behavioural patterns in performing spatial tasks) after increasing the sample size. For other limitations, it is recommended that future studies adopt a gender-balanced approach, utilize diverse spatial visualization tasks and prioritize both qualitative and quantitative aspects. Enhancing the reliability and validity of the collected data with statistical methods can be a useful way to draw more robust conclusions about the generalisability of the results. To ensure accurate and representative data, researchers should implement these measures, which will improve the overall quality of the findings. Furthermore, it would be beneficial for future studies to delve into the potential correlation between the performance of participants in spatial visualization tests and their approach to spatial problem-solving, including the various stages involved. For example, the research could seek to understand how participants' scores on the PSVT - categorized as low, medium, and high performers - relate to their problem-solving proficiency, ranging from novice to competent and expert levels. Finally, this study used Polya's problem-solving steps and novice-to-expert problem-solving schemas as theoretical frameworks. However, alternative theoretical frameworks may produce additional findings on how learners achieve spatial visualisation generalisations.

**Author contributions:** All authors have sufficiently contributed to the study and agreed with the results and conclusions.

**Data availability:** The data supporting this study's findings are available upon request. Interested researchers may contact the corresponding author for access to the data.

**Declaration of interest:** The authors declare that no competing interests exist.

**Ethical statement:** Authors declared that the study was approved by the deanship of the Faculty of Education, Anadolu University on 2 November 2015 with protocol code 1524.

**Funding:** No funding was obtained for this study.

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